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Comments on the Statistical Origin of Black Hole Entropy

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This paper shows that the black hole entropy can be interpreted as emerging as a result of missing information about the exact state of the matter from which the black hole was formed.

KEY WORDS: black hole; entropy.

1. INTRODUCTION

Although 30 years have past since the discovery of the black hole entropy (Bekenstein, 1972, 1973, 1974; Hawking, 1974, 1975) the discussions on its statistical origin is still a rather hot topic in the literature (see Frolov and Novikov, 1998, and references therein). This probably just tells us that we are still missing a correct understanding of this single number.

In these notes I present simple considerations concerning the origin of the black hole entropy. Some of these arguments are rather old and can probably be found in the existing literature. I will consider only the physically most relevant case of the evaporating black hole and avoid discussion of the eternal black hole in which case the considerations presented below fail (at least when applied naively).

2. NONEQUILIBRIUM GAS AND THE ENTROPY OF THE HAWKING RADIATION

Hawking radiation emitted in empty space is far from thermal equilibrium. Nevertheless it possesses entropy and has the Planckian spectrum. Therefore, first I would like to recall how to calculate the entropy of the photon gas in nonequilibrium state. As it is well known the entropy is related with the missing information about the exact state of the system. Let us characterize the ensemble of the photons located

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within a box of volume V by its energy spectrum. In this case we only know that the number of photons with frequencies in the interval ω and $\omega + \Delta\omega$ is equal to ΔN_ω, and the complete information about exact microscopic configuration of the photons is missing. In fact, even if the *energy spectrum* is completely specified, every photon can still have an arbitrary, nonspecified direction of propagation, and can be located in any place inside the box. If we require that the unspecified directions of the photon propagation are restricted by the solid angle ΔO then the number of possible indistinguishable microstates for every photon with energy in between ω and $\omega + \Delta\omega$ is equal to

$$\Delta g_\omega = 2\int \frac{d^3x\, d^3k}{(2\pi)^3} \simeq \frac{1}{4\pi^3} V\omega^2 \Delta O \Delta\omega, \tag{1}$$

where the factor 2 accounts for two possible polarizations of the photons and we use here and everywhere later the Planck units: $c = G = \hbar = k_{\mathrm{B}} = 1$. As follows from simple combinatorics the total number of indistinguishable microstates for ΔN_ω photons is then

$$\Delta G_\omega = \frac{(\Delta g_\omega - 1 + \Delta N_\omega)!}{(\Delta g_\omega - 1)!(\Delta N_\omega)!} \tag{2}$$

and the total number of microstates for *all* photons is, respectively,

$$\Gamma = \prod_\omega \Delta G_\omega. \tag{3}$$

For a given energy spectrum $\{\Delta N_\omega\}$ all these states are equally probable and therefore the entropy can be defined as

$$S = \ln\Gamma = \sum_\omega \ln\Delta G_\omega. \tag{4}$$

Assuming that $\Delta N_\omega, \Delta g_\omega \gg 1$ and using Stirling's formula to approximate the factorials in (2) we get finally

$$S = \frac{V\Delta O}{4\pi^3}\int d\omega\, \omega^2[(n_\omega + 1)\ln(n_\omega + 1) - n_\omega \ln n_\omega], \tag{5}$$

where the occupation numbers $n_\omega \equiv \Delta N_\omega/\Delta g_\omega$ were introduced. Fixing the total energy of the photons gas to be

$$E = \frac{V\Delta O}{4\pi^3}\int d\omega\, \omega^2 n_\omega \tag{6}$$

and extremizing the entropy with respect to energy spectrum we find that the entropy takes its maximal value for the Planckian spectrum

$$n_\omega = \frac{1}{e^{\omega/T} - 1}, \tag{7}$$

where the temperature T is related to the given total energy E, volume V, and solid angle ΔO as

$$E = \frac{\pi}{60} T^4 V \Delta O. \tag{8}$$

In this case the entropy is equal to

$$S = \frac{4}{3} \frac{E}{T}. \tag{9}$$

We would like to stress that this entropy characterizes the missing information about the exact microscopic state of a nonequilibrium gas of photons which can propagate only within solid angle ΔO and have the Planckian spectrum.

Let us turn to Hawking radiation. It has the Planckian spectrum with the temperature

$$T_{\mathrm{H}} = \frac{1}{8\pi M}, \tag{10}$$

and if the black hole emits the energy ΔM, it carries the entropy

$$\Delta S_{\mathrm{H}} = \frac{4}{3} \frac{\Delta M}{T_{\mathrm{H}}}. \tag{11}$$

At the same time the entropy of the black hole decreases by the amount

$$\Delta S_{\mathrm{BH}} = -\frac{\Delta M}{T_{\mathrm{H}}}. \tag{12}$$

Hence when the black hole evaporates the total entropy of the whole system consisting of Hawking radiation and the black hole increases by

$$\Delta S = \Delta S_{\mathrm{H}} + \Delta S_{\mathrm{BH}} = \frac{1}{3} \frac{\Delta M}{T_{\mathrm{H}}}. \tag{13}$$

As it is clear from (9) the entropy of the emitted radiation does not change as long as the radiation propagates in the space since its total energy and temperature remain the same. At first glance this seems in contradiction with the fact that the total volume occupied by the Hawking quanta emitted, for instance, within time interval Δt at the moment $t = 0$, increases as $V \simeq 4\pi t^2 \Delta t$ for large t. However, the indeterminacy of the concrete photon propagation direction characterized by the solid angle

$$\Delta O \simeq 4\pi \frac{\sigma_{\mathrm{BH}}}{4\pi t^2} \tag{14}$$

decreases with time and thus $V \Delta O$ in (8) remains constant in complete agreement with the fact that the temperature does not change (here $\sigma_{\mathrm{BH}} \propto M^2$ is the effective black hole cross section).

In this consideration we neglected the grey factor which arises because the potential barrier outside the black hole disturbs the thermal spectrum. However, if one takes it into account, the main conclusion that the total entropy increases remains unchanged.

3. ON THE ORIGIN OF BLACK HOLE ENTROPY

The idea of a statistical interpretation of the black hole entropy is rather simple and roughly can be formulated as follows: take the emitted Hawking radiation and reverse it forming a new black hole. Then the entropy of this black hole should be about the entropy of the Hawking radiation. Actually if we want to build a black hole from the photons with the total energy M we could first put them in the box of volume $V \sim D^3$, where D is the typical size of the box and then collect these photons inside the volume M^3. How one could do it realistically is not so important for the general question of the origin of the entropy. Could we then identify the maximal possible entropy of these photons with the entropy of the formed black hole? As follows from (8) the entropy gets its maximal value when the photons have a thermal spectrum with temperature

$$T \sim M^{1/4} D^{-3/4} (\Delta O)^{-1/4} \tag{15}$$

and it is equal to

$$S \sim M/T \sim M^{3/4} D^{3/4} (\Delta O)^{1/4}. \tag{16}$$

If one would take the box size D to be of the order of the black hole size $\sim M$ and $\Delta O \sim O(1)$ then $S \sim M^{3/2}$ compared to the required M^2, which is too small to explain the black hole entropy. If $D \to \infty$ then $S \to \infty$. However the size of the box cannot be too large, otherwise the black hole which was already formed would evaporate before we would be able to put all the matter inside it. Since the lifetime of the black hole of mass M is about M^3, the size of the box cannot much exceed $D \sim M^3$. With this value of D the entropy is $S \sim M^3 (\Delta O)^{1/4}$. Most of the photons are located near the border of the box. If the black hole is formed in the center of the box, their directions of propagation cannot be completely arbitrary. The photons have to be directed in such a way to arrive in a small region of size M in the center of the box. This restricts the solid angle ΔO for most of them by

$$\Delta O \sim \frac{M^2}{D^2} \sim \frac{1}{M^4} \tag{17}$$

and the maximal possible entropy of the matter from which we can form the black hole is $S \sim M^2$. The system of mirrors which one could use to redirect the photons complicates the consideration but leaves the conclusion unchanged. Thus this approach opens the way to understand the statistical origin of the black hole entropy as a result of lost information about the exact microscopical initial state of the matter from which the black hole was formed.

It is clear why the entropy of the black hole should not depend on the number of fields. In fact, in the case of N massless fields,

$$M \sim NT^4 D^3 \Delta O. \tag{18}$$

On the other hand the rate of evaporation is proportional to the number of fields and therefore the lifetime of the black hole and respectively the size of the box D should be N times smaller. Taking into account that $\Delta O \sim M^2/D^2$, we see that N is cancelled in the expression (18), and hence the temperature and entropy, $S \sim M/T$, should not depend on the number of fields.

Now we could address the question about the exact numerical coefficient in the formula for the entropy. As already mentioned in the above approach we seem to have just reversed the Hawking flux and explained the black hole entropy via the entropy of the Hawking radiation itself. Therefore one could expect that the result can be 4/3 times bigger than it should be. However the real situation is more complicated. To get a more precise picture of what is going on let us find how much entropy matter can bring into the black hole. Namely, let us take already existing black hole of mass M and given the amount of matter ΔE, ask how much entropy this matter can "add to the black hole" taking into account that in the process of the absorption the black hole continues to evaporate, losing its mass and respectively the entropy. If the matter is absorbed by the black hole within the time interval Δt the black hole emits during this time the energy $L_{\mathrm{H}}\Delta t$, where L_{H} is the total Hawking flux. Therefore the mass of the black hole increases only by amount

$$\Delta M = \Delta E - L_{\mathrm{H}}\Delta t. \tag{19}$$

The photons emitted by the black hole fill a shell of width Δt, the radius of which grows with time as t. These photons have the temperature T_{H} given by (10) and the total amount of energy they carry can be calculated as

$$L_{\mathrm{H}}\Delta t = \frac{\pi}{60} T_{\mathrm{H}}^4 V \Delta O = \frac{\pi^2}{15}\sigma_{\mathrm{BH}} T_{\mathrm{H}}^4 \Delta t, \tag{20}$$

where we took into account that $V = 4\pi t^2 \Delta t$ and used the formula (14) for the solid angle ΔO. The photons which are responsible for the mass increase ΔM bring to the black hole maximal entropy if they are thermal; hence,

$$\Delta M = \frac{\pi^2}{15}\sigma_{\mathrm{BH}} T_*^4 \Delta t, \tag{21}$$

where T_* is the temperature of the photons. *Given* ΔE, one gets from (19)

$$\Delta t = \left(\frac{\pi^2}{15}\sigma_{\mathrm{BH}}\right)^{-1} \frac{\Delta E}{(T_*^4 + T_{\mathrm{H}}^4} \tag{22}$$

and accordingly the amount of entropy which matter bring into the black hole is

$$\Delta S = \frac{4}{3}\frac{\Delta M}{T_*} = \frac{4}{3}\frac{T_*^3}{T_*^4 + T_{\mathrm{H}}^4}\Delta E. \tag{23}$$

This expression takes the maximal value at $T_* = 3^{1/4}T_{\mathrm{H}}$. Hence when the mass of the black hole increases by ΔM, the maximal amount of entropy which matter can bring into black hole is

$$\Delta S = \frac{4}{3^{5/4}}\frac{\Delta M}{T_{\mathrm{H}}} \simeq 1.0131\frac{\Delta M}{T_{\mathrm{H}}}. \tag{24}$$

Within 1% accuracy this result is in agreement with the formula (12). If the interpretation of the black hole entropy presented here is correct then this little mismatch can probably be referred to the neglected influence of the potential barrier on radiation. The exact calculations in this case are rather complicated. However, one can easily see that the "barrier effect" decreases the coefficient in (24), which, at least, "acts in the right direction."

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